MATHEMATICAL DISCOURSE FOR TEACHING: A DISCURSIVE FRAMEWORK FOR ANALYZING PROFESSIONAL DEVELOPMENT

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The framework of mathematical knowledge for teaching (MKT) is brought under the discursive framework of Commognition in order to track learning in professional development (PD). I follow MKT in differentiating between subject matter discourse and pedagogical discourse. The framework, which I call Mathematical Discourse for Teaching (MDT) permits a combined view on mathematical and meta-mathematical issues as constituted in discourse. Such meta-issues are found to be a significant part of what is taught and learned in a particular PD, where mathematics Ph.D. students teach elementary school teachers. Through the analysis of a lesson on parity I show how "knowing" has different meanings in mathematical and pedagogical discourses, and find evidence of learning in the evolving ways in which the parties use this term.

INTRODUCTION

What are teachers learning? This is an important question for any professional development (herein PD) program. Yet it is not clear how we should go about answering it. Though the ultimate goal of PD is a sustainable change in teaching practices, it is important to track learning as it occurs or fails to occur. In this paper I present a discursive framework for conceptualizing and analyzing knowledge and learning in mathematics PD, and demonstrate how this framework helps make sense of a particular session on parity, in which the participants were 1st and 2nd grade teachers and the instructor was a mathematics Ph.D. student. This unusual PD setting highlights the strength of the discursive approach; the instructor and the teachers are shown to have had very different ideas about what it means to know, learn and do mathematics, ideas that are constituted in their discursive practices. The crossover of these meta-mathematical ideas, as mathematical content is being discussed, is shown to be a significant aspect of the learning that is taking place.

THEORETICAL FRAMEWORK

The framework of MKT – Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) has been influential in conceptualizing what mathematics teachers need to know for effective teaching, differentiating between subject matter and pedagogical content knowledge (PCK). However, to track learning as it occurs in PD, we must find indications of learning in the parties' discourse. For this I propose to embed MKT in an overarching discursive framework. The discursive approach I adopt is *commognition*

[1] This research was supported by the Israel Science Foundation (grant No. 615/13).

(Sfard, 2008), whose basic tenet is that fields of human knowledge (such as mathematics) are nothing more than well defined forms of communication, and thus communication and cognition are aspects of a single entity termed *discourse*[2] . Ball et. al. lament that "after two decades of work, the nature of this bridge [PCK bridging content knowledge and the practice of teaching] remains inadequately understood" (p. 3). The commognitive view, seeing PCK and teaching as aspects of a single entity – discourse – may be exactly what is needed.

To understand how MKT may be embedded (and extended) in a commognitive framework, I first present a short exposition of commognitive assumptions and methods. Discourses are types of communication common to particular communities. They are identifiable through four interrelated characteristic features: *keywords*, *visual mediators*, distinctive *routines*, and generally endorsed *narratives*. Most commognitive research to date has focused on mathematical discourses; however in PD we are interested in the discourse of *teaching mathematics*. This discourse makes use of keywords, mediators, routines and narratives of mathematics, but also of *teaching* mathematics, much in the same way as MKT consists of content knowledge and PCK. Thus, each of the MKT categories of knowledge may be redefined as a discourse, calling their union *Mathematical Discourse for Teaching* (MDT). For the purpose of this paper it will be sufficient to distinguish between a mathematical discourse within MDT (paralleling subject matter) and a discourse of teaching mathematics, which I will call Pedagogical Content Discourse (PCD, paralleling PCK). The keywords, mediators, routines and narratives of PCD will be those that are related to teaching, students and curriculum, for example: words such as *difficult, prior knowledge*, *understand*, *misconception*; visual mediators such as manipulatives, routines of teaching, and narratives about how to teach particular content. The notion of *discourse* goes far beyond the cognitivist notion of *knowledge*. To demonstrate this point, the empirical part of this paper analyzes discursive aspects of the notion of *knowing* that some mathematical claim is true. Following Wittgenstein (1958, p. 20), the meaning of a word is taken to be the ways in which it is used, which in our framework means: what are the endorsed narratives in which the word *knowing* features, what are the routines that are invoked by this word, and what are the visual mediators and other keywords associated with it.

METHOD

The PD under investigation was the initiative of a university professor of mathematics, and was taught by mathematics graduate students. Approximately 90 teachers enrolled in the 2011-12 program, which consisted of ten 3-hour sessions taught in six groups spread over the year. The data collected consists of audio recordings of all the sessions, interviews with the instructors before and after the lessons, and teacher questionnaires

[2] The present analysis does not rely strongly on this assumption, and is valid under the weaker assumption that ways of talking do not neutrally reflect social practices such as teaching but rather play an active role in forming them.

– expectations at the outset and feedback after each session. In this paper I analyze part of a lesson on parity in which approximately 15 1st and 2nd grade teachers participated.

The decision to focus on meanings of *knowing* is not arbitrary; rules and routines by which knowledge is endorsed are a central characteristic of mathematical discourse.

The instructors' stated goal for the PD was mathematical – to broaden and deepen the teachers' understanding of the mathematical content they teach. The teachers' expectations, based on questionnaires, were pedagogical – classroom-ready activities and teaching tips. These conflicting goals are the backdrop for my discursive analysis.

DATA ANALYSIS

A comprehensive analysis of the transcript is beyond the scope of this report. I limit my analysis to utterances that reflect meanings of the word know for various participants. I omit utterances that are not relevant for the analysis.

Turns	Duration	What's going on
1-84	4:30	Teachers suggest 5 definitions for even number
85-280	15:00	Discussion: Do we want to give this as a definition?
281-321	3:00	Comparing definitions – which are similar?
322-401	6:30	Even + even = even. How do we know this?
402-480	4:00	How to define an odd number
481-665	11:00	Sign of parity (even ones digit) – why does it work?

Table 1: Overview of transcript data

Segment 1: Do we want to give this as a definition?

85 I[3]: Do we want to give 0, 2, 4, 6, 8, etc as a definition of even number?

91 I: If we tell a child that 0, 2, 4, 6, 8, etc are even, will he know to say if 1024 is an even number?

96 T1: Of course he'll know, according to the ones digit.

98 T2: If we only explain it to him this way.

Here are two different meanings of *knowing*. T1, drawing on the teaching routines of her Pedagogical Content Discourse (PCD), says that children *know* 1024 is even based on a rote endorsement routine (checking ones digit). In contrast, T2 understood the instructor's intention – that the imaginary child only *knows* what he was told explicitly – the *definition* – and that this knowing should be the basis of endorsement.

114 I: What's bothering *me* is that I can continue differently. 0, 2, 4, 6, 8, then 12.

119 T3: But we learned skip counting; he knows it's by 2, he won't pull a 12 on you.

A real child *knows* that 10 follows 8, thus in a pedagogical discourse skipping by 2 does not need to be made explicit. However the instructor's endorsement routine is mathematical in spite of his pedagogical phrasing (*will he know*), where *knowing* is based on what is explicit in the definition.

[3] 'I' indicates *Instructor*. 'T2' (capital T) indicates a particular teacher. t165 (small t) indicates turn number 165.

124 I: So when I skip by 2 from 0 I reach 2, 4, 6, and I can continue ... eventually I'll arrive, it's not very efficient to say if 1024 is even, but it's something.

The instructor "fixes" the definition – skipping by 2 is made explicit; yet it is being judged by a rather basic routine – determining evenness – and is deemed inefficient.

They proceed to discuss the definition: *a number that divides into two identical parts*.

158 T4: But... for example divide 5 into two and a half and two and a half.

163 I: Alright, that's important. In grade 2 and 1, I'm not sure the kids know...

164 T4: They know only halves.

165 I: Ok, if they know then we must be precise.

The instructor seems to have adopted a pedagogical discourse. Mathematically speaking, the precision of a definition does not depend on what any particular audience does or does not know, but an imprecise definition may be endorsed in a pedagogical discourse if the imprecision is unlikely to create a problem for students.

166 T5: They say 5 is divisible, they take the concrete, break the stick...

167 I: Ok... even numbers are only in the context of integers. We don't even know fractions. A number will be called even if I can divide, if I can take that quantity of objects and divide them into two equally large sets.

168 I: This is one way. I'll write another: a number is even if one can take such a quantity of objects and divide them into equally large sets without applying violence, without breaking things along the way. We don't permit breaking.

In the context of integers are the words of a mathematician, who has alternative contexts (natural, integer, rational, real, or complex numbers). "*We don't even know fractions*" is a code, having little to do with what real people know. In retrospect t165 appears less pedagogical. It is not a question of whether children know that a 5-foot stick can be divided equally into 2, but rather are rational numbers part of the children's world? The instructor is now aware of two different discourses. In the pedagogical (t168) we specify *without violence*, since halves are in the child's discourse; in the mathematical (t167) this is not necessary; everything is in the context of integers.

Segment 2: Proving even plus even is even

322 I: Let's say I gave you some oranges, and the number of oranges is divisible by two, that is even. And I also gave you oranges, and you checked, and this number is also even. Now we take the oranges that you both received and put them in a crate. Do I need to check all over if the number is even or not?

329 T6: No. It's even.

330 I: Why?

331 T6: Because it's divisible by two. Even plus even is even.

338 T7: If mine is divisible by 2 and hers is divisible by 2, the definition didn't change... mine remains even and hers remains even, why should it change?

The instructor chooses to ask about the sum of even numbers realized by *quantities*. The teachers return the discussion to abstract numbers. T6 knows that

even+even=even, but this doesn't answer the instructor's *why*. In contrast, T7 accepts the need to prove the claim based on a definition – *divisible by two* – but does not yet see what exactly needs to be proven.

347	I:	You checked, and [each of the quantities] can be arranged in pairs.
350	T8:	You transfer them in pairs, you don't change [the pairing].

The instructor takes them back to *quantities* and T8 completes the mathematical proof.

Segment 3: Definition of odd number

397	T9:	Every odd [number], if I go with a division into pairs, you have one left.
398	I:	Why?... So what's an odd number?

Odd number has not yet been defined. To endorse the narrative in t397 the instructor explicitly asks for a definition , which will become the basis for an endorsement.

437	I:	Suppose I tell you that a number is odd if it's not even. How do we show that the remainder, when we try to divide into pairs, I'll have one left over?
445	T9:	I'd ask them to arrange in pairs... I'd like them to experience it themselves... Because if the remainder is 3, they need to check if this is really the remainder... so they see the two that can be arranged in [another] pair.

The instructor gives his definition for *odd*. Knowing in t437 relies on *showing*, but what does showing mean? T9 suggests a demonstration, using children as a visual mediator. This routine is clearly pedagogical, but it is also mathematical – in t445 this demonstration becomes the foundation for a generic proof by contradiction – if you have 3 left over, you can form another pair.

Segment 4: Sign of parity

481	I:	Let's try to understand now from the definitions we have, why if a number's ones digit is 0, 2, 4, 6, or 8 - it's even. Here's a number...
500	T11:	The ones digit is the end of your pairing. After you've paired them, what you have at the beginning doesn't matter; it's only the bottom line that matters... You bring down the ones digit.
512	T8:	All the numbers before the ones digit are even.

T11 is proving based on a definition (pairs), but has not provided an acceptable argument. She appears to be influenced by the long division routine. T8 provides the missing link – we have already shown that the sum of two even numbers is even. She will show that all numbers are the sum of even numbers and the ones digit. The routine here is proving a property based on previously proven properties; we no longer need to refer all the way back to the definition.

554	T12:	When she says 90 and 500 are even, she's basing it on the ones digit. You must! How else can you know that 1000 is even?
568	T13:	I know it's a multiple of 2. 500 times 2 is 1000.

T12's rote endorsement routine is so entrenched that she can't imagine any other. T13's proof draws on the more abstract definition of even number – multiple of 2. Later, the

instructor helps generalize – numbers are shown to be a sum of an even number (in fact divisible by 10) and the ones digit. Thus, every number is either even+even=even or even+odd=odd.

DISCUSSION

The transcript has shown various *meanings* (in the commognitive sense) of knowing, that is: *narratives* of knowing, *routines* invoked for knowing (e.g. proof), *visual mediators* used to support knowing (e.g. demonstration), and *words* associated with knowing (e.g. *showing*). In Pedagogical Content Discourse (PCD) we see meanings that are concerned with learners and the rules by which they endorse their narratives (t96, t331). In this discourse knowing is not linear – learners may know halves before they have officially learned fractions. Conversely, in the instructor's mathematical discourse knowing is structured. Its endorsement routines begin with definitions, and proceed through theorems that are proven based on these definitions. Furthermore, it is reflective – at any point we know what we "know" and what has yet to be shown. With this in mind, I now ask about the learning that took place, where learning is conceptualized as discursive change. The limited scope of this report cannot show that a learning trajectory was completed. *Opportunities* for learning, where interlocutors meet new discourses and engage in them, will be the focus of this discussion.

Participation in mathematical discourse may be ritualized or explorative (Sfard & Lavie, 2005). The goal of exploration is endorsing new narratives, thus explorative discourse will focus on the autonomous derivation of new narratives and their deductive endorsement. The PD episode can be seen as modeling explorative participation in mathematical discourse, where progressively sophisticated endorsement rules are introduced. This is seen twice. First in the mathematical content where the topic is parity (what are even numbers, prove that even+even=even). In this context, virtually all of the mathematical narratives came from the teachers. The instructor's contribution was in organizing well known narratives into a structure, where endorsement begins with definitions and proceeds, by means of deductive proof, to more sophisticated properties and theorems. The second exploration was meta-mathematical, where the implicit topic was *definition* (“do we want to take this as a definition?”). In both contexts the rules of endorsement evolved. Evenness was first endorsed based on a "rote" property (ones' digit), later it was based on checking a definition, and finally on proven properties. At the meta-mathematical level, definitions were at first endorsed for efficiently deciding if a number is even, later for their productiveness in routines of proving properties and theorems. For the teachers, engaging in these explorative routines is not only a model for classroom teaching, it is also an opportunity to "forget" the rote endorsement routines they have adopted as adults, which for many have become automatic, and recall what there is to learn in such a seemingly straightforward topic as parity.

What in the instructor's pedagogical discourse enabled learning? Modeling, as described above, is not the only tool the instructor used. When an expert is teaching

novices, the expert's discourse may be incomprehensible to the learner, and it is up to the expert to adopt a discourse that bridges the discursive gap[4] . I have shown instances where this is achieved by means of a discursive move called interdiscursivity – "the use of elements in one discourse and social practice which carry… meanings from other discourses and social practices*"* (Candlin & Maley, 1997). A common instructor move was carrying mathematical meanings of words (e.g. *knowing*, *checking*) into pedagogical narratives (t91 t322), all in the context of mathematical routines of proving. The instructor chose to mediate one proof by means of quantities (t347) – a pedagogical realization of *number* – after teachers failed to find a proof using abstract numbers (t331).

Much of the commognitive research to date has focused on the asymmetrical situation of children learning. In PD, adult learners are accomplished teachers, and thus the situation is more symmetrical. It is not only the teachers who learned - the instructor came to appreciate the significance of PCD (e.g. t168). Furthermore, the teachers did not blindly adopt the instructor's patterns of participation in mathematical discourse; the discourse that emerged is an interdiscursive synthesis: t554 prefers a decimal decomposition (1000+500+90+2) over the instructor's decomposition, recognizing place value as a critical topic, and in t445 T9 added a pedagogical mediator – children pairing up – to achieve a mathematical proof. This interdiscursivity on the part of teachers shows that they are appropriating[5] a new mathematical discourse – an indication that learning is taking place. For this to happen, the teachers and the instructor need opportunities to reflect on the mathematics in the context of teaching, thus bridging the gap between their different goals for the PD. The instructor's interdiscursive routines support this. This is also supported in the open nature of the questions he asks, e.g. *do we want to give this as a definition of even number?* Who is meant by *we*? What are the considerations to *want* a particular definition? *Give* to whom? How do we endorse a statement as a *definition*? The fact that all these are left open permits the discussion to draw on multiple discourses. The pedagogical discourse is concerned with learners, for whom numbers are realized as quantities (*a number that can be divided into 2 equal sets*). It addresses classroom routines such as determining efficiently if a number is even. In the mathematical discourse the abstract concept of number is disassociated from *quantity*, precision is crucial, and the routines that involve definitions, such as proving properties, are more sophisticated. The instructor was careful not to let the teachers' pedagogical concerns derail his mathematical goals, but he delayed voicing his own ideas until after the teachers had had their say (t114). It is clear that the instructor was uncomfortable with imprecise phrasing *number that can be divided into quantities*, but he merely revoiced it more precisely – *I can take that quantity of objects*...(t167) – perhaps recognizing that the less precise wording is

[4] This is a discursive paraphrasing of Wertsch's notion of intersubjectivity (1984).

[5] *Appropriation* as used by Moschkovich (2004), in the sense of actively transforming goals and meaning.

productive in a pedagogical context. Even when he adopted parts of a pedagogical discourse (t168), it was alongside the mathematical discourse he is aiming for (t167).

SUMMARY

In this paper I have argued for a discursive approach in analyzing learning in PD, and have shown how a commognitive embedding and extension of mathematical knowledge for teaching, which I call Mathematical Discourse for Teaching (MDT), provides both theoretical framework and methods for such an analysis. This framework highlights discursive aspects of knowing, which may be difficult to conceptualize in a more cognitivist approach. Through focusing on a discursive analysis of meanings of *knowing*, I have shown the kind of learning, conceptualized as discursive change, that is taking place alongside the learning of mathematics. The instructor adopted elements of the teachers' PCD, and the teachers participated in an explorative mathematical discussion, which drew on the instructor's university routines and narratives and on the teachers' pedagogical discourse. In this discussion, the concept of *definition* took on new meanings, as it was used in increasingly sophisticated mathematical endorsement routines. This explorative experience may eventually serve as a model for the teachers' classroom teaching. They did not blindly adopt the instructor's discourse, but rather transformed the mathematical discourse into a discourse *for teaching*, appropriating it for their pedagogical purposes.

In this paper I too have tried to model an explorative discursive practice, enriching the commognitive framework with new words, routines and narratives, interdiscursively drawing on other theories (i.e. discourses) such as MKT.

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